An Axiomatic Study of Scoring Rule Markets

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Prediction markets

**Prediction market**: mechanism wherein agents buy/sell “contracts” . . . thereby revealing “predictions” about a future event.

**Contract**: function $f : \text{outcomes} \rightarrow \text{money}$.

**Question**: How to choose available contracts/prices at each time?
Example

Predict: total number of Trump Tweets in 2018

Contract: pay off 1 cent for every tweet

Cost function: convex $C':$ total contracts sold $\rightarrow$ total cost paid.

If $\theta$ contracts have been sold so far, payment is $C(\theta + 100) - C(\theta)$. 

Current price: $2.56$

I’ll buy 100
Prior work and this paper

Previously studied: cost function markets
- The price converges to *expected value* of the contract
- They are great\(^1\)

Previously proposed generalization: *scoring rule markets*\(^2\) (SRMs)
- Can make *other kinds of predictions*
- But are they great?

This paper:
- Propose *axioms* to address this question,
- apply to e.g. mode, median markets,
- characterize satisfaction of all axioms.

\(^1\)[Abernethy, Chen, Wortman Vaughan 2013]
\(^2\)[Lambert, Pennock, Shoham 2008]
Outline

1 Define scoring rule markets
2 Axioms and key examples
3 Characterization and new market
4 End talk
**Background: Properties of distributions**

Property or statistic of a probability distribution: \( \Gamma : \Delta_Y \rightarrow \mathcal{R} \).

- mean
- mode
- median

**Scoring rule:** function \( S : \mathcal{R} \times Y \rightarrow \mathbb{R} \).

- \( S(r, y) = -(r - y)^2 \) elicits mean
- \( S(r, y) = 1_{r=y} \) elicits mode
- \( S(r, y) = -|r - y| \) elicits median
Why focus on SRMs?

Axiom (Incentive Compatibility for a property)

- market histories $\rightarrow$ prediction $r$
- max utility $\iff$ accurate prediction

Axiom (Path independence)

No gain from making a sequence of trades versus just one.

- Market states
  - Many consecutive trades
  - Single trade
Why focus on SRMs?

**Theorem**

Incentive Compatibility and Path Independence $\Rightarrow$ SRM.

**Definition (SRM$^3$)**

In a scoring rule market (SRM), the net payoff for moving the prediction from $r'$ to $r$ is

$$S(r, y) - S(r', y).$$

$^3$[Hanson 2003; Lambert, Pennock, Shoham 2008]
Robustness for free

**Arbitrage**: purchase of a contract that is profitable in expectation for every belief.

**Theorem**

*All SRMs satisfy no arbitrage*: there is never an arbitrage opportunity.
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Example: Mode

Consider the SRM defined by $S(r, y) = \alpha \mathbb{1}_{r=y}$.

If $\alpha$ is small:

If $\alpha$ is large:
First new axiom

**Liability** from purchasing contract(s): maximum possible net loss.

Axiom (Bounded Trader Budget‘)

Agents can usefully participate while maintaining arbitrarily small liability.

Theorem

No *SRM for any “finite property” can satisfy BTB.*
Example: Median

Consider the SRM defined by \( S(r, y) = -|r - y| \).

Theorem

*If beliefs contain no point masses, every SRM for every quantile property satisfies Bounded Trader Budget.*
Motivating the main axiom

What can you do in a **market**? Both **buy** and **sell**.

- Current price: $2.56
- **I'll sell 30**

But e.g. in the median market, agents sometimes...

- ... cannot decrease **risk** by “selling back” contracts
- ... cannot even decrease **liability**!
Main axioms

**Axiom (Weak Neutralization)**

For any agent with liability $d$, there always exists a trade yielding net liability strictly less than $d$.

$\Rightarrow$ can always reduce liability.

**Axiom (Trade Neutralization)**

For any agent with liability $d$, there always exists a trade yielding constant net liability strictly less than $d$.

$\Rightarrow$ can always reduce liability and eliminate risk.
Example: Median, revisited

Consider the SRM defined by $S(r, y) = -|r - y|$.

Theorem

No SRM for any quantile satisfies Weak Neutralization (nor Trade Neutralization, therefore).
Example: Mean

Theorem (known/direct)

For any expectation of a bounded random variable, there exist SRMs satisfying all axioms.
(In particular, a cost function based market.)
Outline

1. Define scoring rule markets
2. Axioms and key examples
3. Characterization and new market
4. End talk
**Theorem (Main)**

*Any SRM satisfying Trade Neutralization can be written as a cost-function based market.*

*Proof idea:* (1) Lemma showing that contracts mod price form a subgroup of $\mathbb{R}^k$; (2) show pricing is given by single cost function. *(Hidden: bunch of convex analysis.)*

**Corollary (Main)**

*Any market satisfying all our axioms is cost-function based, hence (essentially) *elicits an expectation.*
What about WN? New market idea

**Predict**: ratio of expectations $\frac{\mathbb{E}X}{\mathbb{E}Y}$, e.g. $\frac{\mathbb{E} \text{ Trump Tweets}}{\mathbb{E} \text{ Bieber Tweets}}$.

**Market**: use cost function market for Trump Tweets

**But**: you *pay* in units of “Bieber contracts”

Current price: 0.07

I’ll buy 100

Satisfies WN, but not TN!
Takeaways

- Scoring rule markets for properties like medians, modes, ...
- Proposed axioms for “good” (great?) markets
- Only property to satisfy all axioms: expectations
- Investigation leads to new market design ideas
- Other axioms?
- Innovative prediction mechanism ideas?

Thanks!