UPenn NETS 412: Algorithmic Game Theory
Homework 3

Instructor: Bo Waggoner
Due: by beginning of class, February 22, 2018
Turn in *electronically* via Gradescope.

Problem 1 (6 points)

Consider a 2 person game. If player 1 chooses $a_1 \in A_1$ and player 2 chooses $a_2 \in A_2$, the payoff to player 1 is $u_1(a_1, a_2)$ while the payoff to player 2 is $u_2(a_1, a_2)$. The game is said to have the **conflict property** if for every pair of action profiles a and a', the following is true:

$$u_1(a) < u_1(a') \Rightarrow u_2(a) > u_2(a')$$
$$u_2(a) < u_2(a') \Rightarrow u_1(a) > u_1(a')$$

In other words, whenever player 1 prefers that both players play according to a, player 2 prefers that both players play according to a', and vice versa.

Part a (1 point) Give an example of a non-zero sum game that has the conflict property.

Part b (2 points) Prove that zero sum games have the conflict property.

Part c (3 points) The security level, v_1 of player 1 is

$$v_1 = \max_{a_1 \in A_1} \min_{a_2 \in A_2} u_1(a_1, a_2).$$

The security level of player 2 is

$$v_2 = \max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2).$$

Suppose the game has a pure strategy Nash equilibrium. Show that in this equilibrium player 1 enjoys a payoff of v_1 while player 2 enjoys a payoff of v_2.

Hint. Consider $x^* \in A_1$, the pure strategy that achieves player 1's security level, and $y^* \in A_2$, the pure strategy that achieves the security level for player 2.

Problem 2 (13 points)

Consider the classical so-called "Battle of the Sexes" game, in which the two players each select an activity, with different preferences, and recieve utility for coordinating. *You may want to refer to this problem and the next simultaneously.*

	Football	Opera
Football	$(5, 1)$	$(0, 0)$
Opera	$(0, 0)$	$(1, 5)$

Part a (1 point) Describe **all** dominant strategy Nash equilibria of this game (if any). (No proof required.)

Part b (1 point) Describe **all** pure strategy Nash equilibria of this game (if any). (No proof required.)

Part c (1 point) Desribe **all** mixed-strategy Nash equilibria of this game (if any). (No proof required.)

Part d (2 points) Describe **all** correlated equilibria of this game. Justify your answer.

Part e (4 points) We can measure outcomes of games in several ways to measure if they are "good" or "bad" for the players. One measure is **social welfare**, which is the sum of the player's utilities. What are the maximum and minimum social welfares of any Nash equilibrium? (Recall that "Nash equilibrium" by default refers to mixed-strategy.) What are the maximum and minimum social welfares of any correlated equilibrium?

Part f (4 points) Another measure is the expected utility of the "worst-off" player, i.e. the minimum of the expected utility of player 1 and the expected utility of player 2. What are the maximum and minimum such values in any Nash equilibrium? What are the maximum and minimum such values in any correlated equilibrium?

Problem 3 (8 points)

In the lecture notes, we state (mostly) without proof the following:

> [...] we have so far considered several solution concepts: Dominant strategy equilibria (DSE), Pure strategy Nash equilibria (PSNE), mixed strategy Nash equilibria (MSNE), correlated equilibria (CE), and coarse correlated equilibria (CCE), and we know the following strict containments:
>
> $$DSE \subset PSNE \subset MSNE \subset CE \subset CCE$$

In this problem, we'll construct a proof of these containments.

Part a (2 points) Explain why a Dominant Strategy Nash Equilibrium is a Pure Strategy Nash Equilibrium. That is, why is $DSE \subset PSNE$? Give an example of a 2×2 game which has a PSNE which is *not* a DSE, and identify this equilibrium. This shows that the containment is strict.

Part b (2 points) Explain why a Pure Strategy Nash Equilibrium is a Mixed Strategy Nash Equilibrium. That is, why is $PSNE \subset MSNE$? Give an example of a game which has a MSNE which is *not* a PSNE, and identify this equilibrium. This shows that the containment is strict.

Part c (2 points) Explain why a Mixed Strategy Nash Equilibrium is a Correlated Equilibrium. That is, why is $MSNE \subset CE$? Give an example of a game which has a CE that is *not* a MSNE. This shows that the containment is strict.

Part d (2 points) Explain why a Correlated Equilibrium is also a Coarse Correlated Equilibrium. That is, why is $CE \subset CCE$? Give an example of a game which has a CCE which is *not* a CE, and identify this equilibrium. This shows that the containment is strict.