UPenn NETS 412: Algorithmic Game Theory
Homework 5

Instructor: Bo Waggoner
Due: by beginning of class, April 5, 2018
Turn in *electronically* via Gradescope.

Problem 1 (10 points)

Consider a prediction problem with Prediction Pattie predicting two outcomes, $\mathcal{Y} = \{\text{rain, no rain}\}$. Suppose Pattie believes the probability of rain is $q \in [0, 1]$. Suppose Pattie reports $p \in [0, 1]$ as the probability of rain, and consider the "quadratic" or "Brier" score:

$$S(p, \text{rain}) = 2p - \left(p^2 + (1-p)^2\right)$$
$$S(p, \text{no rain}) = 2(1-p) - \left(p^2 + (1-p)^2\right).$$

(Note here we are writing p and q as numbers in $[0, 1]$ *rather than full probability distributions. This is equivalent because there are only two outcomes.)*

Part a (4 points) Suppose Pattie reports truthfully (recall she believes a probability q of rain, $1 - q$ of no rain).

What is her expected score, as a function of q? Call this $f(q)$. Plot $f(q)$ (a rough hand-drawn plot is fine).

Part b (2 points) Suppose Pattie believes reports $p = 0.6$. Draw on your plot her expected score $S(0.6; q)$ for this report as a function of q? (Or make a new plot including both this function and $f(q)$.)

Hint: You can compute $S(0.6; q)$*, but you will also be able to eyeball it if you understand the lecture notes! Where does this function touch* $f(q)$ *and how does it relate to* $f(q)$*?*

Part c (2 points) Suppose Pattie believes $q = 0.4$. What is her expected score for reporting truthfully? For reporting $p = 0.6$? Give the numbers and identify both of these points on your above plot.

Part d (2 points) Argue that this scoring rule is proper. You can use anything we showed in class, use your plot to make an informal geometric argument, or directly argue using calculations.

Problem 2 (8 points)

Again consider predicting rain or no rain. For $p \in [0,1]$, suppose we have a convex function $f(p)$.

Given f, we can define a scoring rule as follows:

$$S(p, \text{rain}) = f(p) + \frac{df(p)}{dp}(1-p)$$
$$S(p, \text{no rain}) = f(p) + \frac{df(p)}{dp}(0-p).$$

Part a (2 points) Let $f(p) = e^p$. What are $S(p, \text{rain})$ and $S(p, \text{no rain})$?

Part b (2 points) Let $f(p) = 1.5 + p^2$. What are $S(p, \text{rain})$ and $S(p, \text{no rain})$?

Part c (2 points) Suppose you were asked to predict an event. Your belief is that the probability of the event is $q = 0.05$. Which of the two previous scoring rules would you rather is used to score you (and why)? Does your answer change if your belief is $q = 0.95$?

Part d (2 points) Let $f(p) = -p^2$. Now f is not a convex function, but we could still define S as described above. Is S a proper scoring rule? If so, argue why; if not, give a counterexample of why not.

Bonus (1 point): use a plot of f similar to Problem 1 to illustrate your answer. You may find this helpful anyway!

Problem 3 (10 points)

(Peer prediction.) We have two agents, Ashley and Becky. Ashley observes whether it rains in Atlanta, while Becky observes whether it rains in Boston. Each one only observes rain in their own city and does not know what the other observes. We want to design a mechanism that incentivizes them to truthfully report their observations, without the mechanism ever knowing for sure what the ground truth is.

The prior probabilities are as follows: Pr[rain in both] $= 0.2$, Pr[rain in Boston only] $=$ 0.3, Pr[rain in Atlanta only] $= 0.3$, Pr[rain in neither] $= 0.2$.

Part a (3 points) Suppose Ashley observes rain in Atlanta. What is her posterior belief of the probability of rain in Boston? How about if she observes no rain? (show your work)

Part b (3 points) Suppose Becky observes rain in Boston. What is his posterior belief of the probability of rain in Atlanta? How about if he observes no rain? (show your work)

Part c (4 points) Design a mechanism to elicit reports from Ashley and Becky, such that truthful reporting by both is a strict equilibrium (i.e. any deviation from truthfulness gives strictly lower utility).

Your mechanism should take in a report from Ashley which is either "rain" or "no rain" and a report from Becky which is either "rain" or "no rain". It should then assign a payoff to each player.

You can either give the exact numbers of how much to pay each in each scenario (in which case, justify how you got them), or describe how to calculate the payoffs and why this produces a truthful equilibrium.

Suggestion: utilize a strictly proper scoring rule.