UPenn NETS 412: Algorithmic Game Theory
Homework 6

Instructor: Bo Waggoner
Due: 8:59pm, April 22, 2018
Turn in *electronically* via Gradescope.

Problem 1 **(10 points)**

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Each player corresponds to one of the n vertices in V, let us call the players *cities*. Each player has two actions: $A_i = \{0, 1\}$. The action taken by city $i \in V$ is denoted a_i. $a_i = 1$ means that city i decides to build a tennis court and $a_i = 0$ to mean it does not. If it builds a court, it pays a cost c, but enjoys a benefit v. If it declines to build a court, it bears no cost, but still enjoys a benefit, γv. We assume $v > c$ and $v - c < \gamma v$.

If $a = (a_1, a_2, \ldots, a_n)$ is a profile of actions, denote by $u_i(a)$ the utility that city i enjoys. If $a_i = 1$, then $u_i(a) = v - c$. If $a_i = 0$ and there is at least one city j such that $(i, j) \in E$ and $a_j = 1$, then, $u_i(a) = \gamma v$. In all other cases, $u_i(a) = 0$.

In this problem, you only need to consider *pure strategies* and pure strategy equilibria.

Part a (2 points) When is it a best response to build a tennis court? When is it a best response not to build?

Part b (2 points) Describe a pure strategy Nash equilibrium of this game.

Part c (3 points) Define the total welfare of any profile to be the sum of agent utilities in that profile. Given an example of a graph for which the price of anarchy approaches $\frac{v-c}{\gamma v}$ as n, the number of vertices, approaches infinity. (Here we use the utilities definition of price of anarchy, not costs.) What is the price of stability of the game on your graph?

Part d (3 points) What is the best possible price of anarchy, and what graph achieves that value? What is the total welfare?

Problem 2 **(12 points)**

[Spectrum Auctions] Three firms are bidding in an auction for the rights to broadcast over various wireless spectrum frequencies. The firms may acquire one, two, or three of the following spectra, and their valuations in billions of dollars are given in the following table:

Part a (1 point) What is the welfare-maximizing allocation (ignoring prices charged)?

firm, kHz	300	400	500	300,400	400,500	300,500	300,400,500
A	1	1	1	3	3	2	7
B	2	0	0	7	5	3	8
C	0	0	3	0	3	3	6

Table 1: Firm spectrum valuations

Part b (2 points) How much does the VCG mechanism charge each player? What is the revenue of the auction?

Part c (3 points) Suppose that Firm B deviates in an attempt to capture more spectra and reports its valuation falsely instead as the following:

firm, kHz	300	400	500	300,400	400,500	300,500	300,400,500
B	2	0	0	7	5	3	11

Table 2: Firm B's unilateral deviation to misreport

A key property of the VCG is incentive compatibility - that is, players should not be made better off by misreporting their valuations. Calculate the allocation and payments under this deviation, keeping in mind that its true valuations are as above. Show by direct calculation that Firm B is not made strictly better off via this deviation (i.e. incentive compatibility holds).

Part d (6 points) Suppose only 300 kHz and 400 kHz are up for auction, and the valuations are as follows (Table 3).

Suppose that Firm A can secretly communicate with Firm C, and offers Firm C a bribe to collude. The offer is as follows: if both firms falsely report their values as in Table 4, Firm A will pay Firm C \$500MM (that is, 0.5B).

What are the old allocation and prices? What are the new allocation and prices? Is firm C better off? Ignoring ethics, is firm A better off?

firm, KhZ	300	400	300,400
A	2	2	4
B	1	1	5
C	1	1	4

Table 3: Part d: New valuations

firm, KhZ	300	400	300,400
A	6	0	4
C	0	6	4

Table 4: Part d: collusive repotrts

Problem 3 (7 points)

Consider a VCG auction with 10 identical items $\{a_1, ..., a_{10}\}$ and n bidders. It is a closed bid auction, all bidders submit their valuation for the goods and the auctioneer decides the allocation and payments. The auctioneer values the items at zero, and can freely dispose of any extra.

For the following situations, describe how the items will be allocated and the payment of each player following the VCG mechanism.

Part a (2 points) $n = 10$. bidders can only get one item and they report their true values for the good $\{v_1, ..., v_{10}\}$

Part b (2 points) $n = 15$. bidders can only get one item and they report their true values for the good $\{v_1, ..., v_{15}\}$

Part c (3 points) $n = 12$. the fist bidder values the item at 100 and wants exactly one. bidders 2 through 11 value the item at 5 and want exactly one. The final bidder values the item at 10 each but only if she can get all 10 items, otherwise she values them at 0.

Problem 4 (6 points)

Sponsored search is advertising sold at auction where merchants bid for positioning alongside web search results. Advertisers bid for placement on the page in an auction-style format where the larger their bid the more likely their listing will appear above other ads on the page. By convention, advertisers pay *per click*, meaning that they pay only when a user clicks on their ad, and do not pay if their ad is displayed but not clicked.

Let n be the number of bidders and $m < n$ the number of slots. The search engine estimates α_j, the probability that a user will click on the j^{th} slot. The quantity α_j is called a *click through rate* (CTR). It is usually presumed that $\alpha_j \geq \alpha_{j+1}$ for $j = 1, \ldots, m-1$.[1] For each bidder i, let v_i be the value for a click. Assume that bidders are ordered so that $v_1 \geq v_2 \geq \ldots \geq v_n$. If bidder i is matched to slot j and charged p per click, i's utility for that match is $\alpha_j(v_i - p)$.

[1] The assumption that CTR decays monotonically with lower slots is a distinguishing feature of keyword auctions; in particular, it implies that all bidders prefer the first slot to the second, the second slot to the third, etc. The assumption that CTR does not depend on the bidder is for simplicity only.

Part a (2 points) We wish to match bidders to slots so that at most one bidder is matched to each slot and each slot is matched to at most one bidder. Show that assigning bidder i to slot i for $i = 1, \ldots, m$ produces a matching of maximum total value (welfare).

Part b (2 points) In the Generalized Second Price auction used by Google, each bidder i submits a bid, b_i, say. The highest bid is assigned to slot 1, the next highest to slot 2 and so on (ties broken arbitrarily). The bidder assigned to slot 1 is charged the second highest bid per click. The bidder assigned to slot 2 is charged the third highest bid per click and so on. Show that bidding one's value is a not a dominant strategy equilibrium of this auction.

Part c (2 points) Is the Generalized Second Price auction truthful? That is, does it have an equilibrium where all bidders bid truthfully?