UPenn NETS 412: Algorithmic Game Theory	January 16, 2018
Lecture 1	
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Basic Definitions

In this class we introduce some of the basic definitions and notation we will be using throughout the semester.

Definition 1 *A **game** consists of:*

1. *A set of players numbered* $1, \ldots, n$.
2. *A finite set of actions* A_i *for each player* $i = 1, \ldots, n$. *We write* $A = \times_{i=1}^{n} A_i$ *to denote the action space for all players, and* $A_{-i} = \times_{j \neq i} A_j$ *to denote the action space of all players excluding player* j.
3. *A utility function* $u_i : A \to \mathbb{R}$ *for each player* $i = 1, \ldots, n$. *That is,* $u_i(a)$ *denotes the utility of player* i *when all players act according to* a. *We sometimes write this as* $u_i(a_i, a_{-i})$ *to denote* i*'s action* a_i *and the actions of all other players* a_{-i}.

To be more precise, this is a finite, simultaneous-move game, because (respectively) there are a finite number of actions per player and all players select an action simultaneously. We will see other kinds of games later in the course.

The basic assumption in game theory is that players will always try and act so as to maximize their utility. This is well defined when the actions of the other players are fixed:

Definition 2 *A **best-response** for player* i *to a set of actions* $a_{-i} \in A_{-i}$ *is any action* $a_i \in A_i$ *that maximizes* $u_i(a_i, a_{-i})$:

$$a_i \in \arg\max_{a \in A_i} u_i(a, a_{-i})$$

A general idea in game theory is this: "In any stable situation, all players should be playing a best response." (Otherwise, by definition, the situation would not be stable – somebody would want to change their action.)

So we can ask: when are there stable solutions?

Definition 3 *For a player* i, *an action* $a \in A_i$ ***(weakly) dominates*** *action* $a' \in A_i$ *if it is always beneficial to play* a *over* a'. *That is, if for all* $a_{-i} \in A_{-i}$:

$$u_i(a, a_{-i}) \geq u_i(a', a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$.

You can normally eliminate weakly dominated strategies from consideration – there is never a situation in which they are the uniquely optimal best response.

Definition 4 *An action* $a \in A_i$ *is **dominant** for player* i *if it weakly dominates all actions* $a' \neq a \in A_i$.

Note that if an action a is dominant, this is a very strong guarantee – it is always a best response, and so a rational player can safely play it without needing to reason about what her opponents are doing.

Dominant strategies will typically not exist, but when they *do* exist for all players, it is easy to see what rational players should do:

Definition 5 *An action profile* $a = (a_1, \ldots, a_n) \in A$ *is a **dominant strategy equilibrium** of a game if for every* $i = 1, \ldots, n$, a_i *is a dominant strategy for player* i.

	Confess	Silent
Confess	$(1,1)$	$(5,0)$
Silent	$(0,5)$	$(3,3)$

Figure 1: Prisoner's Dilemma. The first player or "row player" selects action Confess (top row) or Silent (bottom row). The "column player" selects Confess (left column) or Silent (right column). The utilities are then described by the square they select, in order (row player utility, column player utility). For example, if the row player Confesses and the column player is Silent, this selects the upper-right square $(5,0)$ where the row player gets utility 5 and the column player gets utility 0. In general, this matrix representation is sometimes called **normal form**.

	Football	Opera
Football	$(5,1)$	$(0,0)$
Opera	$(0,0)$	$(1,5)$

	Heads	Tails
Heads	$(1,-1)$	$(-1,1)$
Tails	$(-1,1)$	$(1,-1)$

Figure 2: Battle of the Sexes and Matching Pennies

Example 1 *(Confess, Confess) is a dominant strategy equilibrium in Prisoner's Dilemma.*

How should we make predictions if a dominant strategy equilibrium does not exist? Even in this case, there may still be *dominated* strategies, which we can remove from consideration. Eliminating these leads to a new, residual game, in which further strategies might be dominated. Continuing in this way can sometimes lead to a unique remaining action profile. This is called "Iterated Elimination of Dominated Strategies".

What if this process doesn't eliminate anything? We can still directly ask for a "stable" profile of actions:

Definition 6 *A profile of actions $a = (a_1, \ldots, a_n) \in A$ is a **pure strategy Nash Equilibrium** if for each player i and for all $a'_i \in A_i$:*

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

This seems like a reasonable solution concept, but does using it as a prediction contradict what we might predict using iterated elimination of dominated strategies? No:

Claim 7 *If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.*

Proof Homework! ■

Unfortunately, pure strategy Nash equilibria are neither guaranteed to exist, nor to be unique when they do exist.

Example 2 *Battle of the sexes has two pure strategy Nash equilibria, and matching pennies has none.*

So how should one predict behavior in a game in which no pure strategy Nash equilibrium exists? Let's take matching pennies as an example: how should you play? (nb: matching pennies is an example of a zero-sum game, which we will study in more depth later):

Definition 8 *A two-player game is **zero-sum** if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)*

In matching pennies (like rock paper scissors) you should randomize to thwart your opponent: the best you can do is to flip a coin and play heads 50% of the time, and tails 50% of the time. Let's give a definition that allows us to reason about randomized strategies like this.

Definition 9 *A* ***mixed-strategy*** $p_i \in \Delta A_i$ *is a probability distribution over actions* $a_i \in A_i$*: i.e. a set of numbers* $p_i(a_i)$ *such that:*

1. $p_i(a_i) \geq 0$ *for all* $a_i \in A_i$
2. $\sum_{a_i \in A_i} p_i(a_i) = 1$.

For $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$, we write:

$$u_i(p) = E_{a_i \sim p_i}[u_i(a)]$$

i.e. we assume that each player draws an action independently from her mixed strategy, and that player i's utility for this randomized set of actions is her expected utility of the realization.

Definition 10 *A* ***mixed strategy Nash equilibrium*** *is a tuple* $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ *such that for all* i*, and for all* $a_i \in A_i$*:*

$$u_i(p_1, p_{-i}) \geq u_i(a_i, p_{-i})$$

Fortunately, these always exist!

Theorem 11 (Nash) *Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.*

(The proof is non-constructive, so its not necessarily clear how to find one of these, even though they exist.)