UPenn NETS 412: Algorithmic Game Theory	April 12, 2018
Lecture 18	
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VCG and Truthfulness in Auctions

Continuing from last time, we will consider the most popular instantiation of the Groves mechanism, "VCG", and see some of its nice properties. Then we will discuss truthfulness in general in auction design, including an important "revelation principle" and Myerson's Lemma for single-parameter environments.

Recall the setup: we have a set of alternatives A and n agents with valuation functions $v_i : A \to \mathbb{R}_{\geq 0}$. A mechanism consists of an allocation rule X and payment rule P. Each takes in the reported valuations $\hat{v} = (\hat{v}_1, \dots, \hat{v}_n)$. They output an alternative $a = X(\hat{v})$ and $p = P(\hat{v})$, where $p \in \mathbb{R}^n$ is the payment vector. We call $o = (a, p)$ the outcome of the mechanism. Agent i's utility is

$$u_i(o) = v_i(a) - p_i.$$

Recall that the Groves mechanism has choice rule $X(v) = \arg\max_{a \in A} \sum_i v_i(a)$; that is, it picks the social-welfare-maximizing alternative, call it a^*. The Groves mechanism has payment rule $P(v)_i = h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*)$ where h_i can be any fixed function of all reports except i's.

Last time, we saw that the Groves mechanism is dominant strategy incentive compatible and has a price of stability equal to 1, i.e. maximizes social welfare. However, depending on the choice of h_i, it may not satisfy no deficit: the auctioneer may lose money.

First, some notation. Suppose we fix a set of reports v. Let $a^* = \arg\max_{a \in A} \sum_i v_i(a)$, the welfare-maximizing alternative. Let $W = \sum_i v_i(a^*)$, the welfare under this alternative. Now suppose we remove bidder i and recalculate these values. Let $a^*_{-i} = \arg\max_{a \in A} \sum_{j \neq i} v_j(a)$ and let $W_{-i} = \sum_{j \neq i} v_j(a^*_{-i})$.

Definition 1 *The* ***Vickrey-Clarke-Groves (VCG)*** *mechanism has choice rule:*

$$X(v) = a^*$$

and payment rule

$$P(v)_i = W_{-i} - \sum_{j \neq i} v_j(a^*).$$

In other words, it is the Groves mechanism with $h_i(v) = W_{-i}$.

The idea behind VCG payments is that every agent i is charged the "negative externality" that she imposes on the market. We take the optimal social welfare of all $j \neq i$ in the world where i didn't exist – that's W_{-i} – and subtract the social welfare of all $j \neq i$ in the real world – that's $\sum_{j \neq i} v_j(a^*)$. This gives the difference between everyone else's welfare that i imposes; i pays exactly this amount. We will show that the VCG mechanism satisfies all of our desiderata.

Theorem 2 *The VCG mechanism dominant strategy incentive compatible and has price of stability* 1.

Proof It is an instantiation of the Groves mechanism. ■

Theorem 3 *The VCG mechanism is individually rational.*

Proof We need to show that, if i participates truthfully, then her utility is at least 0. Her utility is

$$\begin{aligned} u_i(o) &= v_i(a^*) - \left(W_{-i} - \sum_{j \neq i} v_j(a^*) \right) \\ &= \sum_{j=1}^{n} v_j(a^*) \ - \ W_{-i} \\ &= W - W_{-i}. \end{aligned}$$

Now we claim that the optimal social welfare, W, is larger than the optimal social welfare without i, W_{-i}.

$$
\begin{aligned}
W &= \sum_j v_j(a^*) \\
&\geq \sum_j v_j(a^*_{-i}) && \text{by definition of } a^* \\
&\geq \sum_{j \neq i} v_j(a^*_{-i}) && \text{because } v_i(a) \geq 0 \\
&= W_{-i}.
\end{aligned}
$$

■

Finally, to complete the picture:

Theorem 4 *The VCG mechanism is no-deficit.*

Proof We will in fact show the stronger claim that every agent's payment is nonnegative, i.e. $P(v)_i \geq 0$. Recall that:

$$P(v)_i = W_{-i} - \sum_{j \neq i} v_j(a^*).$$

But by definition of a^*_{-i}, we have

$$
\begin{aligned}
\sum_{j \neq i} v_j(a^*) &\leq \sum_{j \neq i} v_j(a^*_{-i}) \\
&= W_{-i}.
\end{aligned}
$$

■

So the VCG mechanism satisfies all of our wildest dreams, in an extremely general setting! Perhaps we can end our study of auction design here?

Not quite. There are two main drawbacks of VCG. First, it only optimizes the objective of social welfare. (Although it can be modified to deal with some similar objectives.) So if we have another goal, such as revenue, our work is not yet done. Second, it is not always computationally efficient!

Fact 5 *In general settings, calculating a^* is NP-hard, therefore, so is running VCG.*

One way to see computational difficulty is to consider a combinatorial auction where there are m items for sale and any subset of the items can be given to any bidder. In this case, each bidder's valuation function must specify her value for each of the 2^m subsets – an exponentially large amount of information! This idea can be extended to an NP-hardness proof.

A final potential issue with VCG is its price of anarchy, rather than its price of stability. We know that its price of anarchy can be arbitrarily small, because the second-price auction is a special case. We may come back to this in a future lecture.