UPenn NETS 412: Algorithmic Game Theory	April 19, 2018
Lecture 20	
Lecturer: Bo Waggoner	*Scribe: Bo Waggoner*

Smoothness and Auction Price of Anarchy

So far, we have seen auctions with very good price of stability, in fact, they achieve optimal social welfare. However, we have not shown any bounds on price of anarchy, and in fact, we saw that the second-price auction has a price of anarchy of zero. (More precisely, it cannot guarantee a price of anarchy strictly above zero.) Since it is a special case of the VCG mechanism, this implies that VCG also cannot guarantee a strictly positive price of anarchy.

On the other hand, we will now show that the *first price* auction actually does have a price of anarchy guarantee.

The first price auction. We have a single item for sale, and each bidder has a value $v_i \in \mathbb{R}_{\geq 0}$ for the item. In the first price auction, each bidder simultaneously submits a "bid" $b_i \in \mathbb{R}_{\geq 0}$. The one with the largest bid wins the item and pays her bid. (Ties are broken arbitrarily.)

Let OPT be the optimal social welfare – in this case, $\text{OPT} = \max_i v_i$.

Lemma 1 ("Smoothness") *For each player i and any pure strategy profile $b = (b_1, \dots, b_n)$,*

$$u_i(b'_i, b_{-i}) + R(b) \geq \frac{1}{2} v_i.$$

Proof First, notice that $R(b) \geq 0$ and $u_i(b'_i, b_{-i}) \geq 0$, because bids are nonnegative and i cannot have negative utility when bidding b'_i. So all we have to show is that $\max\{u_i(b'_i, b_{-i}), R(b)\} \geq \frac{1}{2} v_i$.

If $R(b) \geq \frac{1}{2} v_i$, then we are done. So suppose $R(b) < \frac{1}{2} v_i$. Then when i deviates to $b'_i = \frac{1}{2} v_i$, she is the highest bidder and wins the item! So $u_i(b'_i, b_{-i}) = v_i - p_i = v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$. ■

Theorem 2 *The price of anarchy of the first price auction is at least $\frac{1}{2}$.*

Proof Let $i^* = \arg\max_i v_i$, that is, i has the highest value and $\text{OPT} = v_{i^*}$. Note that strategies may be randomized, so we consider expected welfare in the first-price auction. We can write expected social welfare as follows, where the expectation is over the randomness in b as each bidder plays from her mixed strategy.

$$\begin{aligned} \mathbb{E}\,\text{welfare} &= \mathbb{E}\left[\left(\sum_i u_i(b)\right) + R(b)\right] \\ &\geq \mathbb{E}\left[u_{i^*}(b) + R(b)\right] \\ &\geq \mathbb{E}\left[u_{i^*}(b'_i, b_{-i}) + R(b)\right] \end{aligned}$$

because b is an equilibrium, so each agent's utility is higher playing b_i than by deviating to b'_i. Now by the smoothness lemma, for each pure strategy profile b that we get inside the expectation:

$$\begin{aligned} u_{i^*}(b'_i, b_{-i}) + R(b) &\geq \frac{1}{2} v_{i^*} \\ &= \frac{1}{2}\text{OPT} \end{aligned}$$

So the expected welfare is also at least half of OPT. ■