UPenn NETS 412: Algorithmic Game Theory
Game Theory Practice

Problem 1

		Clyde	
		Silent	Confess
Bonnie	Silent	$-1, -1$	$-10, 0$
	Confess	$0, -10$	$-5, -5$

This game is called 'Prisoner's Dilemma'. Bonnie and Clyde have been arrested on suspicion of bank robbery, but the police do not have enough evidence to convict them without a confession. Bonnie and Clyde are separately brought to different interview rooms and asked whether or not they want to sign a confession.

The police have offered them a deal, where if one confesses and the other does not, the one confessing can walk away with no time in prison and the other will serve a sentence of 10 years. If they both confess, they can split the sentence and each serve 5 years. However, Bonnie and Clyde know that the police don't have enough evidence without a confession, so if they both stay silent, they'll each only serve one year in prison on minor charges.

Part a If Bonnie and Clyde weren't being interviewed separately, and could work together, which outcome would they pick?

Solution. They would both stay silent, as this is the welfare-maximizing outcome.

Part b Suppose that Clyde has already been interviewed and has stayed silent, and Bonnie knows this. What should Bonnie do to maximize her own utility? Why?

Solution. She should confess. Confessing gives her 0 utility while staying silent gives her -1.

Part c Now suppose that Clyde has already been interviewed and has confessed, and Bonnie knows this. What should Bonnie do to maximize her own utility? Why?

Solution. She should confess. The payoff to staying silent is -10 compared to -5 for confessing.

Part d Does Bonnie have a dominant strategy here?

Solution. Yes. Confess is a best-response to both of Clyde's actions, so it is a dominant strategy.

Part e Does Clyde have a dominant strategy here?

Solution. Yes, by a symmetric argument.

Part f What is the Nash equilibrium of this game?

Solution. Both players confess.

Part g Why aren't there any other Nash equilibria?

Solution. Both players are playing a dominant strategy – it is a dominant strategy equilibrium. No other strategy can be a best response, so no other strategy can be an equilibrium.

Problem 2

		Bob	
		Heads	Tails
Alice	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

This game is called 'Matching Pennies' (sometimes 'Odds-and-Evens'). Alice and Bob each have a penny, and they can choose to either show heads or tails. If they both show the same thing (the pennies 'match'), Alice wins. If they show different things, Bob wins.

Part a Suppose Alice knows that Bob is going to show heads. What should she do?

Solution. Show heads. She wins if they match.

Part b Suppose Alice knows that Bob is going to show tails. What should she do?

Solution. Show tails. Again, she wins if they match.

Part c Suppose Bob knows that Alice is going to show heads. What should he do?

Solution. Show tails. He wins if they don't match.

Part d Suppose Bob knows that Alice is going to show tails. What should he do?

Solution. Show heads. Again, she wins if they don't match.

Part e Does either player have a strictly dominant strategy?

Solution. No. For both players, both actions are sometimes the best response to some action of the other player.

Part f Is there a pure strategy Nash equilibrium?

Solution. No. There is no pair of actions where both players are best-responding to each other. In other words, in any one of the outcomes, one of the players would have rather played the other choice.

Part g Suppose instead of playing a strategy deterministically, each player randomizes. Suppose that Alice plays heads with probability p and Bob plays heads with probability q. Suppose that Alice knows that Bob has set $q = .75$. What should she set p to to maximize her utility. In other words, what is Alice's best-response to Bob's choice of $q = .75$?

Solution. Alice should always play heads. If she does, she expects to win with probability 3/4. If she picks something smaller, say $p = .9$, she would win with probability $.9 * .75 + .1 * .25$, which is less than .75.

Part h Suppose Alice knows that Bob has picked $q = .5$. What should she set p to to maximize her utility. In other words, what is Alice's best-response to Bob's choice of $q = .5$?

Solution. Every choice of $p \in [0, 1]$ is a best-response to $q = .5$. This is because regardless of her choice, her expected utility is zero. If she sets $p = p^*$, she expects to win with probability $p^*.5 + (1 - p^*) * .5 = .5$.

Part i If Bob has set $q = .5$, why does Alice picking $p = .9$ not result in the game being in a Nash equilibrium?

Solution. Even though .9 is a best-response to .5, Bob's choice of .5 is not his best-response to Alice picking .9 (he would rather pick $q = 0$). A Nash equilibrium requires that every player is best-responding to every other player.

Part j Why is Alice picking $p = .5$ and Bob picking $q = .5$ a Nash equilibrium?

Solution. We saw earlier for Alice that any choice of p is a best-response to $q = .5$ and by symmetry, any choice of q for Bob is a best-response to Alice picking $p = .5$. Since both players are best-responding to each other, this is a Nash equilibrium.

Part k How do we know that this Nash equilibrium is unique?

Solution. For no other pair of values of p and q are Alice and Bob best-responding to each other. If, for example, p is a little larger than .5, Bob would rather always play tails (i.e. set $q = 0$), but then Alice's best-response would be to always play tails (set $p = 0$), and now we've moved back into pure strategy territory, and we argued earlier that there is no pure strategy equilibrium in this game. Symmetric arguments for perturbations of p and q away from .5 in other directions complete the argument.

Problem 3

Both games we've seen have had a unique equilibrium, but equilibria do not have to be unique. Here we'll analyze a game with both pure and mixed strategy equilibria.

		Bob	
		Party	Movie
Alice	Party	$3, 2$	$1, 1$
	Movie	$0, 0$	$2, 3$

This game is traditionally referred to as 'Battle of the Sexes', but we'll call it 'Coordinating Activities'. Alice and Bob previously discussed going to either their friend's party or to see a movie, but neither can remember what they agreed on. Alice would rather go to the party than to the movie, and Bob would rather see the movie than go to the party, but they'd both rather do the same activity than end up at separate places.

Part a Explain why both players going to the movie or both players going to the party are both pure strategy equilibria.

Solution. At (movie,movie) and (party, party), both players are best-responding to each other. Equivalently, at these profiles, neither player wants to deviate.

Part b Explain why both players randomizing over the two choices, each choosing movie with probability .5 is not a mixed strategy Nash equilibrium.

Solution. If Alice is going to the movie with probability .5, then Bob's expected payoff for going to the movie is 1.5 and his expected payoff for going to the party is 1, so Bob's best response in this case is to always go to the movie, rather than mixing with probability .5.

Part c Explain why each player randomizing by picking their preferred activity with probability .75 is a mixed strategy Nash equilibrium.

Solution. Suppose Alice picks party with probability .75, then Bob's expected payoff for going to the party is $.75(2) + 0.25(0) = 1.5$ and his expected payoff for going to the movie is $0.75(1) + 0.25(3) = 1.5$. Since these are equal, Bob is indifferent between the choices, so any choice of a mix, including going to the movie with probability .75, is a best-response. A symmetric calculation shows that Alice is indifferent when Bob goes to the movie with probability .75, so when both players pick their preferred option with probability .75, they are indeed best-responding to each other, hence this is a Nash equilibrium.

Part d Are there any other mixed strategy Nash equilibria?

Solution. No, for the same reason as in Matching Pennies. For any other choice of probability for Alice, Bob will be better off playing one of his actions deterministically, and symmetrically for Alice. This has to be the only mixed strategy Nash equilibrium.

Problem 4

Here's a game from class.

	Violin	Harmonica
Guitar	(3,-3)	(-1,1)
Drums	(-2,2)	(1,-1)

Part a What is the column player's best response to Guitar? to Drums?

Solution. If the row player plays Guitar, the column player gets -3 from Violin and 1 from Harmonica, so Harmonica. Similarly, if the row player plays Drums, the column player's best response is Violin.

Part b What is the column player's best response if the row player is playing $p = (0.5, 0.5)$, that is, probability 0.5 on Guitar and 0.5 on Drums?

Solution. If the column player plays Violin, she gets expected utility $0.5(-3)+0.5(2) = -0.5$. If she plays Harmonica, she gets $0.5(1) + 0.5(-1) = 0$. So Harmonica is the best response (putting probability 1 on Harmonica).

Part c This game only has one mixed Nash equilibrium. Find it!

Hint: Remember that if a player is randomizing in equilibrium, then she is indifferent between her actions, i.e. they have the same payoff.

Solution. See Lecture 06.