UPenn NETS 412: Algorithmic Game Theory
Midterm Practice Problems 2

Instructor: Bo Waggoner

Problem 1

Consider a game with two players with action profiles $A_1 = A_2 = \{1, ..., 10\}$. The players get the same payoff with the function $u(s_1, s_2) = (s_1 + s_2)/2$ if $s_1 \neq s_2$ and $u(s_1, s_2) = s_1 + 1$ if $s_1 = s_2$. Basically, the function will average the two values given, but give a bonus for giving the same numbers.

Part a List all the pure strategy nash equilibria. Justify your answer.

Part b Find a mixed strategy Nash equilibrium that is not a pure strategy equilibrium.

Part c Give a correlated equilibrium that is not a pure or mixed strategy equilibrium.

Part d Consider the distribution $\mathcal{D}$ where, for each $j = 7, 8, 9, 10$, both players play j with probability $\frac{1}{4}$. Is this a correlated equilibrium? Why or why not?

Part e Consider the above distribution again. Is it a coarse correlated equilibrium? Why or why not?

Part f Now consider the same game except the payoff if $s_1 = s_2$ is $s_1 - .5$. List all the pure strategy nash equilibria. Justify your answer briefly.

Problem 2

You may consult the definition of the polynomial weights algorithm while solving this practice problem.

Part a Suppose all losses are either 0 or 1, and furthermore, suppose you know there will be some perfect action that always gets loss 0 (however, you don't know which action.) What value would you choose for the parameter ϵ in the algorithm? Justify your answer. What algorithm for following expert advice does this remind you of?

Part b Suppose we set ϵ very small, for example, $\frac{1}{T}$. In a sentence or two: What would the practical difference be in how the PW algorithm behaves and why might it perform worse in some settings? (Your answer can be high-level, not involving calculations.)

Part c Suppose we set ϵ large, for example, 0.1 regardless of T. In a sentence or two: What would the practical difference be in how the PW algorithm behaves and why might it perform worse in some settings? (Your answer can be high-level, not involving calculations.)

Problem 3

In this game, each player wants to send flow along a shared channel of maximum capacity 1. On the one hand, each player wants to send as much flow as possible along the channel. On the other hand, the channel becomes less useful the closer it gets to its maximum capacity. Each player can choose to send an amount of flow $x_i \in [0, 1]$ along the channel. (That is, the action set for each player i is $A_i = [0, 1]$, and hence is not finite).

Let $x_i \in [0, 1]$ denote the action of player i. For a profile of actions $x \in A$, let $S = \sum_{j=1}^{n} x_i$ be the total flow. Then player i has utility $u_i(x_i, x_{-i}) = x_i(1 - S)$.

Note that if the total flow is larger than one, each player has negative utility!

Part a Show that this game is a potential game with potential function

$$\phi(x) = -S + \frac{1}{2}S^2 + \frac{1}{2}\sum_{j=1}^{n} x_j^2.$$

Hint: One approach is to consider $\frac{du_i}{dx_i}$ *and* $\frac{d\phi}{dx_i}$.

Part b Find a Nash equilibrium of this game. What is the social welfare at this equilibrium? (*i.e.* the sum of utilities of all the players.)

Part c What is the optimal social welfare? (*i.e.* what is the social welfare at the profile of actions that maximizes it, regardless of whether or not this profile is an equilibrium.) Intuitively, why isn't this optimum achieved in equilibrium?

Problem 4

Consider a game in which there are two firms producing identical goods. Let q_1 and q_2 represent the quantity of the good produced by each firm. The price at which the goods can be sold depends on the quantity produced. The price that the firms can charge for their goods is given as $p(q_1, q + 2) = a - (q_1 + q_2)$, where a is some constant. Finally, each firm incurs a cost of c for each unit of the good produced. The firms experience no fixed costs, so the cost of producing a quantity of zero is zero. We'll assume $c < a$ and that the good is infinitely divisible, so it makes sense to talk about fractional quantities.

This is called a Cournot game (also Cournot duopoly, or Cournot oligopoloy for more than two players). The model was developed by Cournot, a mathematician who lived in the 19th century, and Cournot published his model over a century before the emergence of the field of game theory. This model and its variants are heavily studied in economics.

Part a Write down the profit function of Firm 1 π_1 as a function of c, q_1, q_2, a, where *profit* is the revenue of Firm 1 minus the cost of producing the quantity q_1 of the good.

Part b Suppose Firm 2 has fixed its quantity at $\overline{q}_2$. What quantity q_1 should Firm 1 pick to maximize profit? This is Firm 1's best-response to $\overline{q}_2$.

Hint: One way to figure this out is to take a derivative of the profit function.

Part c By symmetry, Firm 2's best-response function to a choice $\overline{q}_1$ by Firm 1 is identical to that of Firm 1, just with the indices swapped. Find the values of q_1 and q_2 at the Nash equilibrium (in this game, there is a unique, pure strategy Nash equilibrium).

Part d What is the market price at the Nash equilibrium?

Part e How much profit does each firm make?

Part f Suppose the firms instead colluded and each produced $q_i = \frac{a-c}{4}$. What would the market price be? What is the profit of each firm?

Part g In this collusive case, we see that the profit for each firm is higher than in the Nash equilibrium. Explain why this collusive agreement is unsustainable. That is, why is this collusive agreement not a Nash equilibrium?