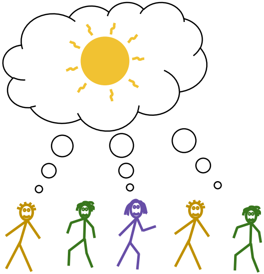
An Axiomatic Study of Scoring Rule Markets



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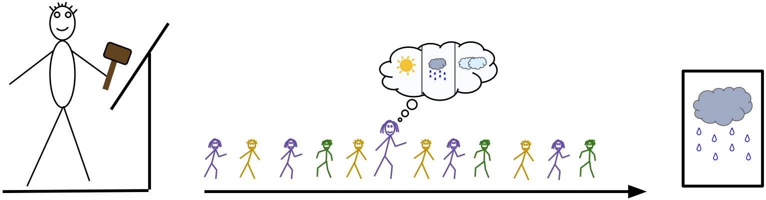
Prediction markets

Prediction market: mechanism wherein agents buy/sell "contracts"
…thereby revealing "predictions" about a future event.

Contract: function f : outcomes $\to$ money.

Question: How to choose available contracts/prices at each time?



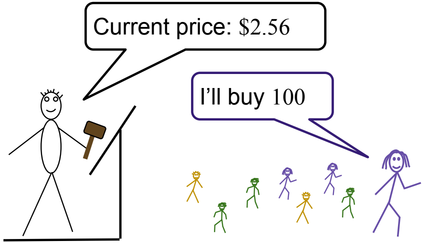
Example

Predict: total number of Trump Tweets in 2018

Contract: pay off 1 cent for every tweet

Cost function: convex C : total contracts sold $\rightarrow$ total cost paid.

If θ contracts have been sold so far, payment is $C(\theta + 100) - C(\theta)$.



Prior work and this paper

Previously studied: cost function markets

- The price converges to *expected value* of the contract
- They are great[1]

Previously proposed generalization: **scoring rule markets**[2] (SRMs)

- Can make *other kinds of predictions*
- But are they great?

This paper:

- Propose **axioms** to address this question,
- apply to e.g. mode, median markets,
- characterize satisfaction of all axioms.

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[1] [*Abernethy, Chen, Wortman Vaughan 2013*]
[2] [*Lambert, Pennock, Shoham 2008*]

Outline

1. **Define scoring rule markets**
2. Axioms and key examples
3. Characterization and new market
4. End talk

Background: Properties of distributions

Property or *statistic* of a probability distribution: $\Gamma : \Delta_{\mathcal{Y}} \to \mathcal{R}$.

- mean
- mode
- median

Scoring rule: function $S : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}$.

- $S(r, y) = -(r - y)^2$ *elicits mean*
- $S(r, y) = \mathbb{1}_{r=y}$ *elicits mode*
- $S(r, y) = -|r - y|$ *elicits median*

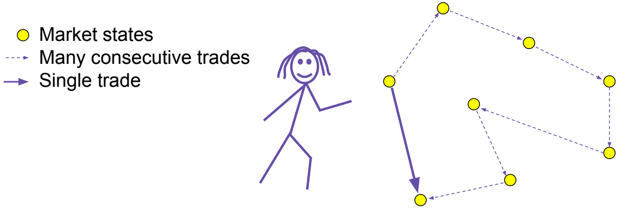
Why focus on SRMs?

Axiom (Incentive Compatibility for a property)

- market histories $\longrightarrow$ prediction r
- max utility $\Longleftrightarrow$ accurate prediction

Axiom (Path independence)

No gain from making a sequence of trades versus just one.



Why focus on SRMs?

Theorem

Incentive Compatibility *and* **Path Independence** $\Rightarrow$ *SRM.*

Definition (SRM[3])

In a **scoring rule market (SRM)**, the net payoff for moving the prediction from r' to r is

$$S(r, y) - S(r', y).$$

[3] [*Hanson 2003; Lambert, Pennock, Shoham 2008*]

Robustness for free

Arbitrage: purchase of a contract that is profitable in expectation for *every* belief.

Theorem

All SRMs satisfy **no arbitrage***: there is never an arbitrage opportunity.*

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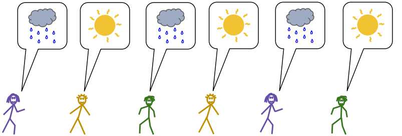
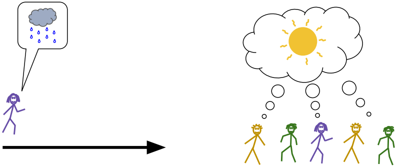
Example: Mode

Consider the SRM defined by $S(r, y) = \alpha \mathbb{1}_{r=y}$.

If α is small:



If α is large:



First new axiom

Liability from purchasing contract(s): maximum possible net loss.

Axiom (Bounded Trader Budget‘)

Agents can usefully participate while maintaining arbitrarily small liability.

Theorem

No *SRM for* **any** *“finite property” can satisfy BTB.*

Example: Median

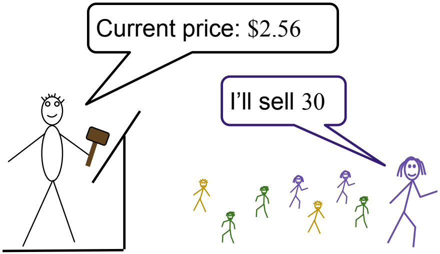
Consider the SRM defined by $S(r, y) = -|r - y|$.

Theorem

If beliefs contain no point masses, **every** *SRM for* **every** *quantile property satisfies Bounded Trader Budget.*

Motivating the main axiom

What can you do in a **market**? Both **buy** and **sell**.



But e.g. in the median market, agents sometimes…

- …cannot decrease **risk** by "selling back" contracts
- …cannot even decrease **liability**!

Main axioms

Axiom (Weak Neutralization)

For any agent with liability d, there always exists a trade yielding net liability strictly less than d.

$\Rightarrow$ *can always reduce liability.*

Axiom (Trade Neutralization)

For any agent with liability d, there always exists a trade yielding **constant** net liability strictly less than d.

$\Rightarrow$ *can always reduce liability and eliminate risk.*

Example: Median, revisited

Consider the SRM defined by $S(r, y) = -|r - y|$.

Theorem

No *SRM for* **any quantile** *satisfies Weak Neutralization (nor Trade Neutralization, therefore).*

Example: Mean

Theorem (known/direct)

For any expectation of a bounded random variable, there exist SRMs satisfying all axioms.

(In particular, a cost function based market.)

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Characterization

Theorem (Main)

Any SRM satisfying Trade Neutralization can be written as a cost-function based market.

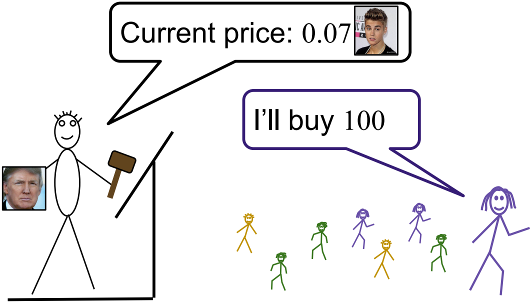
Proof idea: (1) Lemma showing that contracts mod price form a subgroup of $\mathbb{R}^k$; (2) show pricing is given by single cost function.
(Hidden: bunch of convex analysis.)

Corollary (Main)

Any market satisfying all our axioms is cost-function based, hence (essentially) **elicits an expectation**.

What about WN? New market idea

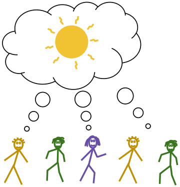
Predict: ratio of expectations $\mathbb{E}X/\mathbb{E}Y$, e.g. $\frac{\mathbb{E}\text{ Trump Tweets}}{\mathbb{E}\text{ Bieber Tweets}}$.
Market: use cost function market for Trump Tweets
But: you *pay* in units of "Bieber contracts"



Satisfies WN, but not TN!

Takeaways

- Scoring rule markets for properties like medians, modes, ...
- Proposed axioms for "good" (great?) markets
- Only property to satisfy all axioms: expectations
- Investigation leads to new market design ideas
- Other axioms?
- Innovative prediction mechanism ideas?



Thanks!